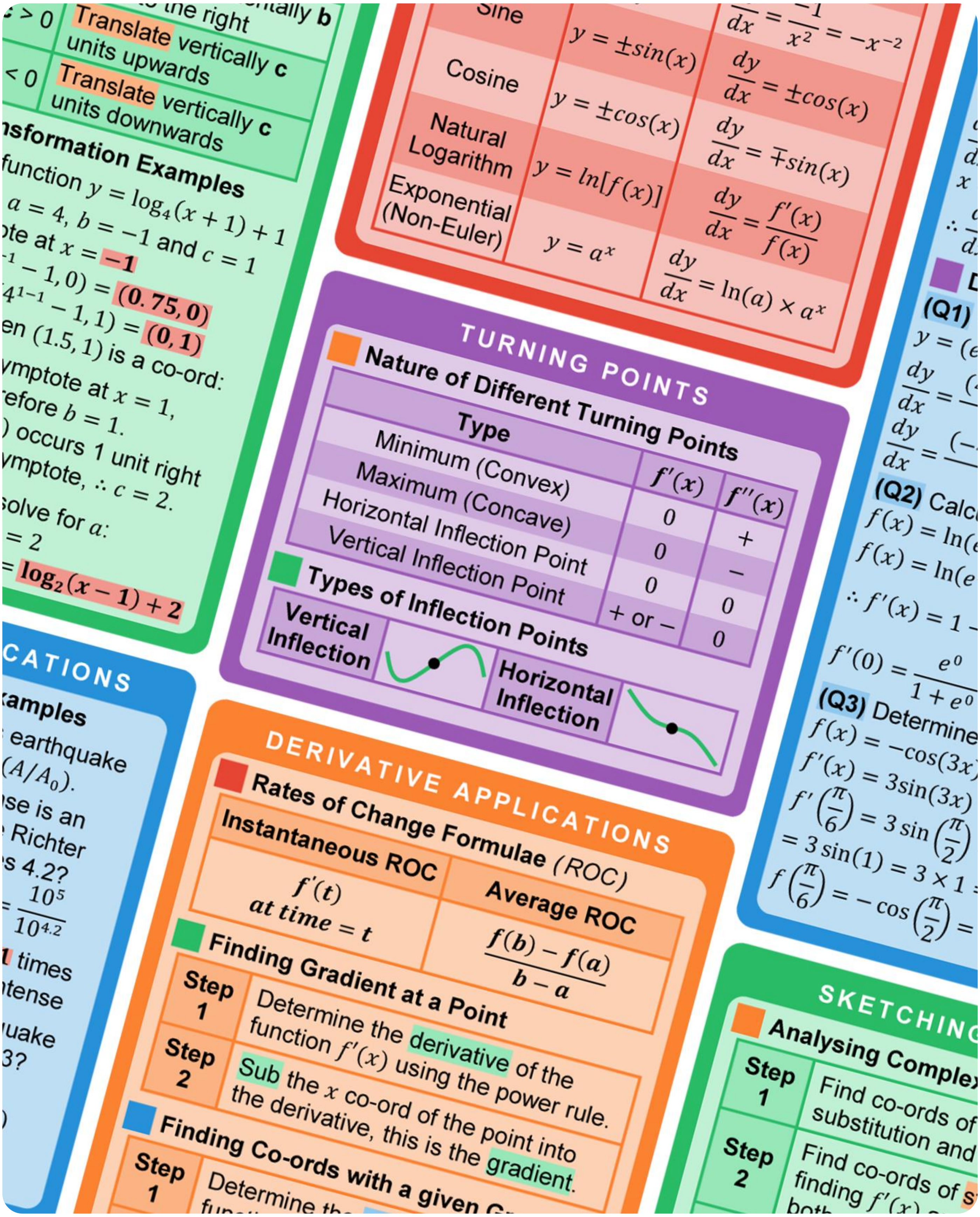

ATAR Mathematics Methods Units 3 & 4

Exam Notes for Western Australian Year 12 Students

Created by Anthony Bochrinis
Version 2.0 (Updated 23/02/19)

Sharpened®

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A Sharpened® Resource

► About the Creator - Anthony Bochrinis

Hello! My name is Anthony and I graduated from high school in 2012, completed a Bachelor of Actuarial Science in 2015, completed my Graduate Diploma in Secondary Education in 2017 and am now a secondary mathematics teacher!

My original exam notes (created in 2013) were inspired by Severus Snape's copy of Advanced Potion Making in Harry Potter and the Half-Blood Prince; a textbook filled with annotations containing all of the pro tips and secrets to help gain a clearer understanding.

Thank you for being a part of my journey in realising that teaching is my lifelong vocation!

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INDICES AND SURDS

INDEX AND SURD LAWS

Index Laws

$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{m\times n}$	$a^0 = 1$
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$	$(ab)^m = a^m \times b^m$
$a^{-m} = \dfrac{1}{a^m}$	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Surd Laws

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$	$\sqrt{a} \times \sqrt{a} = a$
$\sqrt{a} \div \sqrt{b} = \sqrt{\dfrac{a}{b}}$	$m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

Rationalising a Surd

- Removes surd in denominator of a fraction.

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times 1 = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

LOGARITHMS

LOGARITHM LAWS

The Concept of the Logarithm *(logs)*

- The power to which a number (*i.e.* base) must be raised to produce a given number.

$$a^x = y \to x = log_a(y)$$

• a : base number • x : exponent • y : solution	If $2^3 = 8$ then the matching logarithm is $log_2 8 = 3$

Logarithm Laws

- Adding and Subtracting Logarithms:

$log_a(x) + log_a(y) = log_a(x \times y)$	$log_a(x) - log_a(y) = log_a(x \div y)$

- Index Laws of Logarithms:

$log_a(x)^n = n log_a(x)$	$log_a\left(\dfrac{1}{x}\right) = -log_a(x)$
$log_a(a^x) = x$	$a^{\log_a x} = x$

- Logarithm Special Cases:

$log_a(1) = 0$	$log_a(a) = 1$
$log_a(0)$ Cannot exist	$log_a(negative)$ Cannot exist

- Changing Logarithm Base (*i.e.* from B to A):

$$log_a(x) = log_b(x) \div log_b(a)$$

Natural Logarithm and Euler's Number

$$e^x = y \to x = log_e(y) = ln(y)$$

- e : Euler's number (*i.e.* $e = 2.71828...$)
- $ln(x)$: Natural logarithm of x

- Derivation of Euler's Number via Limits:

$\lim\limits_{n\to 0}\left(1+\dfrac{1}{n}\right)^n = e$	$\lim\limits_{n\to 0}\left(1-\dfrac{1}{n}\right)^n = \dfrac{1}{e}$

- Natural Logarithm Limit equations:

$\lim\limits_{h\to 0}\left(\dfrac{a^h-1}{h}\right) = ln(a)$	$\lim\limits_{h\to 0}\left(\dfrac{e^h-1}{h}\right) = ln(e) = 1$

LOGARITHM ALGEBRA

Evaluating Logarithm Examples

(Q1) Evaluate $3\log_2 6 - \log_2 27$

$= \log_2 216 - \log_2 27 = \log_2\left(\dfrac{216}{27}\right) = \log_2 8 = \mathbf{3}$

(Q2) Evaluate $1.5\log_8 4 + 3\log_8 64 - \log_8 1$

$= \log_8(\sqrt{4})^3 + (3\times 2) - 0 = \log_8 8 + 6 = \mathbf{7}$

(Q3) Evaluate $(log135 - log5)/log3^2$

$= \dfrac{log27}{log3^2} = \dfrac{log27}{2log3} = \dfrac{log3^3}{2log3} = \dfrac{3log3}{2log3} = \dfrac{3}{2} = \mathbf{1.5}$

Simplifying Logarithm Examples

(Q1) If $\log_a 5 = p$ and $\log_a 2 = q$, express $\log_a 80a$ in terms of p and q or both.

$\log_a 80a = \log_a(16 \times 5 \times a) = \log_a(2^4 \times 5 \times a)$
$= 4\log_a 2 + \log_a 5 + \log_a a = \mathbf{4q + p + 1}$

(Q2) If $\log_2 5 = x$ and $\log_2 3 = y$, express $\log_2 0.12$ in terms of x and y or both.

$\log_2\left(\dfrac{12}{100}\right) = \log_2\left(\dfrac{3}{25}\right) = \log_2 3 - \log_2 25$
$= \log_2 3 - \log_2 5^2 = \log_2 3 - 2\log_2 5 = \boldsymbol{y - 2x}$

Solving Logarithm Examples

(Q1) Solve for x: $2^{3x-1} = 7 \times 5^{2x}$

$(3x-1)log2 = log(7 \times 5^{2x})$ *Take log of both sides
$3xlog2 - log2 = log7 + 2xlog5$
$3xlog2 - 2xlog5 = log7 + log2$
$x(3log2 + 2log5) = log7 + log2$ *Factorise
$x = \dfrac{log7 + log2}{3log2 + 2log5} = \dfrac{log14}{log8 - log25} = \dfrac{\mathbf{log14}}{\mathbf{log(8/25)}}$

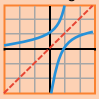
(Q2) Solve for x: $\ln(4x - 2) = -1$

$4x - 1 = e^{-1}, 4x = 2 + \dfrac{1}{e}, \therefore x = \dfrac{\mathbf{1}}{\mathbf{2}} + \dfrac{\mathbf{1}}{\mathbf{4e}}$

INVERSE FUNCTIONS

Inverse Functions

- Exponential function (*i.e.* $y = a^x$) is inverse of the logarithmic function (*i.e.* $y = \log_a x$).



- When inverse functions are plotted together, they are symmetrical about a 45° line (*i.e.* the function $y = x$).

Determining the Inverse of a Function

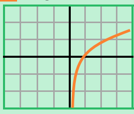
Step 1	Rearrange the function to make x the subject instead of y.
Step 2	Swap the variables x and y, this is the inverse function, $f^{-1}(x)$.

Exponential vs. Logarithmic Function

Let $y = a^x$ $log_a y = x\,log_a a$
$log_a y = \log_a a^x$ $log_a y = x \therefore y = log_a x$

LOGARITHMIC FUNCTION

Logarithmic Function Transformations



Logarithmic
$y = log_a(x - b) + c$
Domain=$\{x \in \mathbb{R}: x > 0\}$
Range=$\{y \in \mathbb{R}\}$

- Important logarithmic function features:

Asymptote	Important Co-ordinates	
Vertical: $x = b$	x-intercept: $(a^{-c} + b, 0)$	Another point: $(a^{1-c} + b, 1)$

- Impact on changing function variables:

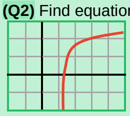
Variable	Condition and Description	
b Adds b to all x-values	$b > 0$	Translate horizontally b units to the left
	$b < 0$	Translate horizontally b units to the right
c Adds c to all y-values	$c > 0$	Translate vertically c units upwards
	$c < 0$	Translate vertically c units downwards

Function Transformation Examples

(Q1) Describe the function $y = \log_4(x+1) + 1$

- From equation, $a = 4$, $b = -1$ and $c = 1$
- Vertical asymptote at $x = \mathbf{-1}$
- x-intercept at $(4^{-1} - 1, 0) = \mathbf{(0.75, 0)}$
- Another point at $(4^{1-1} - 1, 1) = \mathbf{(0, 1)}$

(Q2) Find equation given $(1.5, 1)$ is a co-ord:



- Asymptote at $x = 1$, therefore $b = 1$.
- $(2, 2)$ occurs 1 unit right of asymptote, $\therefore c = 2$.
- Using point $(1.5, 1)$ to solve for a:

$1 = log_a(1.5 - 1) + 2$ $\therefore a = 2$
$-1 = log_a(0.5)$ $\therefore y = \mathbf{log_2(x-1) + 2}$

LOGARITHM APPLICATIONS

Logarithm Applications Examples

(Q1) Richter Scale, R, measures earthquake intensity, A, according to $R = log(A/A_0)$.

(Q1a) How many times more intense is an earthquake that measures 5 on the Richter Scale compared to on that measures 4.2?

$5 = log\left(\dfrac{A_5}{A_0}\right) \to \dfrac{A_5}{A_0} = 10^5$ $\dfrac{A_5}{A_{4.2}} = \dfrac{10^5}{10^{4.2}}$

$4.2 = log\left(\dfrac{A_{4.2}}{A_0}\right) \to \dfrac{A_{4.2}}{A_0} = 10^{4.2}$ $= \mathbf{6.21}$ times more intense

(Q1b) What is the measure of an earthquake that three times stronger than a reading of 6.3?

$R = log(A/A_0)$ $R = log(3A/A_0)$
$6.3 = log(A/A_0)$ $R = log(3) + log(A/A_0)$
$10^{6.3} = A/A_0$ $R = 0.477 + log(10^{6.3})$
*Sub into rule: $R = 0.477 + 6.3 = \mathbf{6.78}$

TRIGONOMETRY

TRIGONOMETRIC ALGEBRA

Unit Circle Formulae

$sin(-x) = -sin(x)$	$cos(-x) = cos(x)$	$tan(-x) = -tan(x)$
$tan(x) = \dfrac{sin(x)}{cos(x)}$	$sin^2(x) + cos^2(x) = 1$	

Trigonometric Identities

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(a)\cos(a)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$
$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$$

Exact Values of Trigonometric Ratios

Deg.	0°	30°	45°	60°	90°
Rad.	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
Sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
Cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
Tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	N/A

DIFFERENTIATION

DERIVATIVE LAWS

Differentiation by First Principles

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

Derivative Laws

Type	Equation	1st Derivative
Product Rule	$y = uv$	$\dfrac{dy}{dx} = u'v + uv'$
Quotient Rule	$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$
Chain Rule	$y = [f(x)]^n$	$\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$
Chain Leibniz	$x = f(t)$ $y = f(t)$	$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dx}{dt}$

Expressing Derivatives

Notation	1st Derivative	2nd Derivative
$y = \cdots$	$dy/dx = \cdots$	$d^2y/dx^2 = \cdots$
$f(x) = \cdots$	$f'(x) = \cdots$	$f''(x) = \cdots$

Common Functions and Derivatives

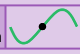
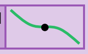
Function	Equation	1st Derivative
Polynomial	$y = ax^n$	$\dfrac{dy}{dx} = n \times ax^{n-1}$
Exponential (Euler)	$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x) \times e^{f(x)}$
Reciprocal	$y = \dfrac{1}{x} = x^{-1}$	$\dfrac{dy}{dx} = \dfrac{-1}{x^2} = -x^{-2}$
Sine	$y = \pm sin(x)$	$\dfrac{dy}{dx} = \pm cos(x)$
Cosine	$y = \pm cos(x)$	$\dfrac{dy}{dx} = \mp sin(x)$
Natural Logarithm	$y = ln[f(x)]$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
Exponential (Non-Euler)	$y = a^x$	$\dfrac{dy}{dx} = \ln(a) \times a^x$

TURNING POINTS

Nature of Different Turning Points

Type	$f'(x)$	$f''(x)$
Minimum (Convex)	0	+
Maximum (Concave)	0	–
Horizontal Inflection Point	0	0
Vertical Inflection Point	+ or –	0

Types of Inflection Points

Vertical Inflection		Horizontal Inflection	

DERIVATIVE APPLICATIONS

Rates of Change Formulae *(ROC)*

Instantaneous ROC	Average ROC
$f'(t)$ at time $= t$	$\dfrac{f(b) - f(a)}{b - a}$

Finding Gradient at a Point

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Sub the x co-ord of the point into the derivative, this is the gradient.

Finding Co-ords with a given Gradient

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Make the given gradient equal to the derivative and solve for x.
Step 3	Sub the x co-ord found in step 2 into the original equation to find the y co-ord, present answer as (x, y).

Co-ords of a Stationary Point

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Make derivative equal to 0 and solve for x (*note*: can be more than one answer when solving).
Step 3	Sub the x co-ord found in step 2 into the original equation to find the y co-ord, present answer as (x, y).

Equation of the Tangent at a Point

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Sub x co-ord of the point into the derivative, this is m in $y = mx + c$.
Step 3	Sub m found in step 2 and x/y co-ord into $y = mx + c$ and solve for c.
Step 4	Write $y = mx + c$ using m from step 2 and c from step 3.

DERIVATIVE ALGEBRA

Product/Quotient/Chain Rule Examples

(Q1) Find $f'(x)$ given $f(x) = 5x(1 - 2x^2)^4$
$f'(x) = (5)(1 - 2x^2)^4 + (5x)(4(1 - 2x^2)^3(-4x))$
$f'(x) = \mathbf{5(1 - 2x^2)^4 - 20x^2(1 - 2x^2)^3}$

(Q2) Find $f'(x)$ given $f(x) = e^{-x}sinx$
$f'(x) = e^{-x}cosx - e^{-x}sinx = \mathbf{e^{-x}(cosx - sinx)}$

(Q3) Find $f'(x)$ given $f(x) = tanx$
$f(x) = \dfrac{sinx}{cosx} \therefore f'(x) = \dfrac{\cos^2 x - sin^2 x}{\cos^2 x} = \dfrac{\mathbf{1}}{\mathbf{cos^2 x}}$

(Q4) Find $f'(x)$ given $f(x) = \sin x/(e^{-x})$
$f'(x) = \dfrac{e^{-x}cosx - -e^{-x}sinx}{(e^{-x})^2} = \dfrac{\mathbf{cosx + sinx}}{\mathbf{e^{-x}}}$

(Q5) Find $f'(x)$ given $f(x) = \ln(x/(x^2 + 1))$
$f(x) = \ln(x) - \ln(x^2 + 1) \therefore f'(x) = \dfrac{\mathbf{1}}{\mathbf{x}} - \dfrac{\mathbf{2x}}{\mathbf{x^2 + 1}}$

(Q6) Find $f'(x)$ given $f(x) = \sqrt{x^4 - x}$
$f(x) = (x^4 - 2x)^{\frac{1}{2}} \therefore f'(x) = \dfrac{\mathbf{(x^4 - x)^{-\frac{1}{2}}}}{\mathbf{2}}\mathbf{(4x^3 - 1)}$

(Q7) Find $f'(x)$ given $f(x) = \log_3(x^3 - 2x)$
$f(x) = \dfrac{\ln(x^3 - 2x)}{\ln(3)} \therefore f'(x) = \dfrac{\mathbf{3x^2 - 2}}{\mathbf{\ln(3)\,(x^3 - 2x)}}$

(Q8) Find $f'(x)$ given $f(x) = \sin^2(5x)$
$f(x) = (\sin(5x))^2 \therefore f'(x) = \mathbf{2(sin5x)(5cos5x)}$

(Q9) Find dy/dx given $y = 2f(4x - 1)$
$dy/dx = 2f'(4x - 1)(4) = \mathbf{8f'(4x - 1)}$

(Q10) Find dy/dx given $y = 6^x$ *Sub y into eq.
$ln(y) = xln(6)$ $dy/dx = \ln(6)\,e^{xln(6)}$
$e^{ln(y)} = e^{xln(6)}$ $dy/dx = \ln(6) \times y$
$y = e^{xln(6)}$ $dy/dx = \ln(6) \times 6^x = \mathbf{6^x ln(6)}$

(Q11) If $x = 4t, y = t^3 - 2$, determine dy/dx in terms of x only and simplify your answer.
$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$ $\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = 3t^2 \times \dfrac{1}{4}$
$x = 4t, \therefore t = x/4$
$\therefore \dfrac{dt}{dx} = \dfrac{1}{4}, \dfrac{dy}{dt} = 3t^2$ $\dfrac{dy}{dx} = \dfrac{3t^2}{4} = \dfrac{3\left(\frac{x}{4}\right)^2}{4} = \dfrac{\mathbf{3x^2}}{\mathbf{64}}$

Derivative Application Examples

(Q1) Calculate the gradient of the function $y = (e^{-2x})/(4x)$ at the point where $x = -1$
$\dfrac{dy}{dx} = \dfrac{(4x)(-2e^{-2x}) - (4)(e^{-2x})}{16x^2}$ *Sub & simplify $x = -1$
$\dfrac{dy}{dx} = \dfrac{(-4)(-2e^2) - (4)(e^2)}{16} = \dfrac{4e^2}{16} = \dfrac{\mathbf{e^2}}{\mathbf{4}}$

(Q2) Calculate the gradient of the function $f(x) = \ln(e^x/(1 + e^x))$ at the point where $x = 0$
$f(x) = \ln(e^x) - \ln(1 + e^x) = x - \ln(1 + e^x)$
$\therefore f'(x) = 1 - \dfrac{e^x}{1 + e^x} = \dfrac{(1 + e^x)(e^x)}{1 + e^x} = \dfrac{e^x}{1 + e^x}$
$f'(0) = \dfrac{e^0}{1 + e^0} = \dfrac{1}{1 + 1} = \dfrac{\mathbf{1}}{\mathbf{2}}$ *Sub & simplify $x = 0$

(Q3) Determine the equation of the tangent of $f(x) = -\cos(3x)$ at the point where $x = \pi/6$
$f'(x) = 3\sin(3x)$ $= -\cos(0) = 0$
$f'\left(\dfrac{\pi}{6}\right) = 3\sin\left(\dfrac{\pi}{2}\right) = 3$ *Sub into $y = 3x + c$
$= 3\sin(1) = 3 \times 1 = 3$ $0 = 3\left(\dfrac{\pi}{6}\right) + c, \therefore c = -\dfrac{\pi}{2}$
$f\left(\dfrac{\pi}{6}\right) = -\cos\left(\dfrac{\pi}{2}\right) = 0$ $\therefore \boldsymbol{y = 3x - \dfrac{\pi}{2}}$

SKETCHING FUNCTIONS

Analysing Complex Functions

Step 1	Find co-ords of x and y intercepts by substitution and factorisation.
Step 2	Find co-ords of stationary points by finding $f'(x)$ and $f''(x)$ and solving both equations for when it equals 0.
Step 3	Find the nature of each turning point by subbing into 2nd derivative.
Step 4	Find long term behaviour for y values as x tends toward $\pm\infty$.

Sketching Functions Example

(Q1) Sketch $f(x) = 2x^3 + 6x^2$ over $-3 \le x \le 3$

- Finding all x and y intercept co-ords:

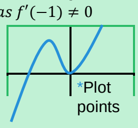
$y = 2(0)^3 + 6(0)^2 = 0, \therefore y - int\ (0,0)$
$0 = 2x^2(x + 3)x = 0, -3 \therefore x - int\ (0,0), (-3,0)$

- Finding location and nature of turning points:

1st derivative: 2nd derivative:
$f'(x) = 6x^2 + 12x$ $f''(x) = 12x + 12$
$0 = 6x^2 + 12x = 6x(x + 2)$ $0 = 12x + 12$
$x = 0, -2$ $x = -1$

- $f(0) = 0, f''(0) > 0 \therefore min\ at\ (0,0)$
- $f(-2) = 8, f''(-2) < 0 \therefore max\ at\ (-2,8)$
- $f(-1) = 4, f''(-1) = 0 \therefore vertical\ point\ of\ inflection\ at\ (-1,4)\ as\ f'(-1) \ne 0$

- Long term behaviour as x tends toward $\pm\infty$:
$x \to +\infty, y \to +\infty$
$x \to -\infty, y \to -\infty$



*Plot points

ATAR Math Methods
Units 3 & 4 Exam Notes

INCREMENTAL FORMULA

Small Change and Approximation

- Calculates the approximate change in a dependent variable y from a small change in the matching independent variable x.

$\delta y \approx \frac{dy}{dx} \times \delta x$	$\frac{\delta y}{y} \approx \frac{dy}{dx} \times \frac{\delta x}{x}$

- δy or δx : small change in x or y (must be small for an accurate approximation).
- $\delta y/y$ or $\delta x/x$: % change in x or y.

Incremental Formula Examples

(Q1) Find the change in y when x changes from 3 to 2.98 in the equation: $y = 3x^2 - 2x$

$\delta y \approx dy/dx \times \delta x$

$\delta y \approx (6x - 2) \times \delta x$ $\qquad \frac{dy}{dx} = 6x - 2$

$\delta y \approx (6(3) - 2) \times (-0.02)$

$\delta y \approx -0.32$ ∴ decrease by **0.32**

(Q2) Radius of a sphere increases from $15cm$ to $15.1cm$, what is the increase in surface area?

$\delta S \approx dS/dr \times \delta r$

$\delta S \approx (8\pi r) \times \delta r$ $\qquad S = 4\pi r^2 \rightarrow \frac{dS}{dr} = 8\pi r$

$\delta S \approx (8\pi(15)) \times (0.01)$

$\delta S \approx 3.77$ ∴ increase by **$3.77cm^2$**

(Q3) Find the change in y when x changes from 1 to 1.1 in the equation: $y = sin(2x) + e^{3x}$

$\delta y \approx dy/dx \times \delta x$

$\delta y \approx (2\cos 2x + 3e^{3x}) \times \delta x$ $\qquad \frac{dy}{dx} = 2\cos(2x) + 3e^{3x}$

$\delta y \approx 2\cos(2) + 3e^{3(1)} \times (0.1)$

$\delta y \approx 5.94$ ∴ increase by **5.94**

(Q4) The radius of a sphere increases by 2%, find the percentage increase in the volume.

$V = \frac{4}{3}\pi r^3$ $\qquad \delta V \approx 4\pi r^2 \times \delta r$ $\qquad \frac{\delta V}{V} \approx 3 \times \frac{\delta r}{r}$

$\frac{dV}{dr} = 4\pi r^2$ $\qquad \frac{\delta V}{V} \approx \frac{4\pi r^2 \delta r}{V}$ $\qquad \frac{\delta V}{V} \approx 3 \times 2\%$

$\frac{\delta V}{\delta r} \approx \frac{dV}{dr}$ $\qquad \frac{\delta V}{V} \approx \frac{4\pi r^2 \delta r}{4\pi r^3/3}$ $\qquad \frac{\delta V}{V} \approx 3 \times 6\%$

$\delta V \approx \frac{dV}{dr} \times \delta r$ $\qquad \frac{\delta V}{V} \approx \frac{3\delta r}{r}$ $\qquad$ ∴ **6%** increase

GROWTH AND DECAY

Growth and Decay Formulae

$A = A_0 e^{kt}$	$\frac{dA}{dt} = kA_0 e^{kt} = kA$

- A_0 : Initial (starting) amount at time = 0.
- k : constant of proportionality.
  - $k > 0$: represents exponential growth.
  - $k < 0$: represents exponential decay.
- t : time (units differ as per the question).

Half Life and Doubling Time

- Half Life: decay specific (for $k < 0$).

$A = 0.5A_0$	Time for initial amount to reduce by 50% (halve).

- Doubling Time: growth specific (for $k > 0$).

$A = 2A_0$	Time for initial amount to increase by 100% (double).

Derivation of Growth/Decay Formulae

$\frac{dA}{dt} = kA$ (i.e. rate is in direct proportion with k).

$dA = kA \times dt$ $\qquad ln(A) = kt + c$

$\frac{dA}{A} = k\, dt$ $\qquad e^{ln(A)} = e^{kt+c}$ $\qquad$ *Let constant $e^c = A_0$

$\int \frac{1}{A} dA = \int k dt$ $\qquad A = e^{kt} \times e^c$

$A = e^{kt} \times A_0$

$A = A_0 e^{kt}$

Growth/Decay Examples

(Q1) Population of 10000 bacteria is decaying according to time measured in minutes after 7am. The time taken for the population to decrease to half its original size is 7 minutes.

(Q1a) Find the constant of proportionality, k.

$A = 0.5A_0$ $\qquad 0.5 = e^{7k}$ $\qquad k = ln(0.5)/7$

∴ $0.5A_0 = A_0 e^{7k}$ $\qquad ln(0.5) = 7k$ $\qquad k =$ **-0.99**

(Q1b) Find the population at 7:05am.

$A = 10000e^{-0.99t} \rightarrow A = 10000e^{-0.99(5)} =$ **6095**

(Q1c) When will the population fall below 100?

$100 = 10000e^{-0.99t} \rightarrow t = 46.507 =$ **46m 31s**

(Q1d) What is the rate of change at 7:15am?

$\frac{dA}{dt} = kA = kA_0 e^{kt} = -0.99 \times 10000e^{-0.99\times15}$

$=$ **-224** bacteria per minute (i.e. decreasing).

(Q2) If $dA/dt = 0.252A$, find the initial value for A given that amount at time = 10 is 565.

$565 = A_0 e^{0.252(10)}$ $\qquad ln(565) - ln(A_0) = 2.52$

$\frac{565}{A_0} = e^{0.252(10)}$ $\qquad ln(A_0) = ln(565) - 2.52$

$ln(A_0) = 3.8168$

$ln\left(\frac{565}{A_0}\right) = 2.52$ $\qquad$ ∴ $A_0 = e^{3.8168} =$ **45.46**

(Q3) The foam in a glass of soft drink shrinks according to $H = 20e^{-0.005t}$ where H is height of the foam in mm and t is time in seconds.

(Q3a) Find the average rate of change of the foam height during the second minute.

$\frac{H(120) - H(60)}{120 - 60} = \frac{10.98 - 14.82}{60} =$ **$-0.064mm$**

(Q3b) Find the instantaneous rate of change of the height of the foam after 24 seconds.

$\frac{dH}{dt} = -0.1e^{-0.005t} \rightarrow \frac{dH}{dt} =$ **$-0.89mm/s$**

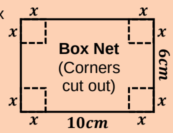
OPTIMISATION

Optimising Dimensions of a Scenario

Step 1	Draw a diagram of the scenario and define all variables.
Step 2	If there are more than 2 variables, reduce the number of variables to 2 by substitution and simplification.
Step 3	Determine the derivative of $f(x)$. ClassPad: $diff(f(x))$
Step 4	Make derivative equal to 0 and solve for x to find turning points. ClassPad: $solve[diff(f(x))] = 0$
Step 5	Find nature of all turning points by subbing in x co-ord found in step 4 by using the second derivative test. ClassPad: $diff[diff(f(x))]$
Step 6	Find optimal dimensions and maximum or minimum value required according to question.

Optimisation Examples

(Q1) A rectangular box is made from a sheet of metal with squares of length x to be cut from the corners and folded. If the sheet of metal is 6cm wide and 10cm long, find x that maximises the volume.



- Identify all equations relevant to question:

$V = lwh \rightarrow 4$ variables in this equation.

- Reduce to two variables by substitution:

$l = 10 - 2x$, $w = 6 - 2x$ and $h = x$

- Find derivative and test all turning points:

$V = (10 - 2x)(6 - 2x)x = 60x - 32x^2 + 4x^3$

$\frac{dV}{dx} = 12x^2 - 64x + 60, \frac{d^2V}{dx^2} = 24x - 64$

Solving for when $dv/dx = 0$: $x = 4.12, 1.21$

$When\ x = 4.12, d^2V/dx^2 = 34.88$ ∴ $minimum$

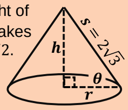
$When\ x = 1.21, d^2V/dx^2 = -34.96$ ∴ $maximum$

- Find dimensions and maximum volume:

Sub $x = 1.21$ to find max $V = 32.84cm^3$

∴ The volume is a max when $x =$ **$1.21cm$**.

(Q2) A cone has a slant height of $2\sqrt{3}\ cm$. The sloped edge makes an angle θ where $0 < \theta < \pi/2$. Find the cone max volume.



- Identify all equations:

$V = \frac{1}{3}\pi r^2 h,\ h = 2\sqrt{3}sin\theta,\ r = 2\sqrt{3}cos\theta$

- Reduce to two variables by substitution:

$V = \frac{\pi(2\sqrt{3}cos\theta)^2(2\sqrt{3}sin\theta)}{3} = 12\sqrt{3}cos^2\theta sin\theta$

$V = 12\sqrt{3}(1 - sin^2\theta)sin\theta = 12\sqrt{3}(sin\theta - sin^3\theta)$

- Find derivative and test turning point:

$dV/dx = 12\sqrt{3}cos\theta - 36\sqrt{3}sin^2\theta cos\theta$

Solving for when $dV/d\theta = 0$: $x = 0, 0.6155$

Disregard $\theta = 0$ as a possible solution,

$When\ \theta = 0.6155, d^2V/d\theta^2 = -48$ ∴ $maximum$

- Find dimensions and maximum volume:

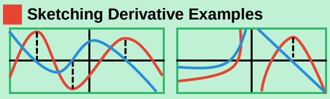
Sub $\theta = 0.6155$ to find max $V =$ **$16.76cm^3$**

SKETCHING DERIVATIVES

Sketching Derivative Functions

- All local max/min are x-intercepts on $f'(x)$.
- All points where the function is increasing, $f'(x)$ is above the x-axis and vice versa.
- Where there is a point of inflection on the graph (vertical or horizontal), the derivative has a maximum or minimum turning point.

Sketching Derivative Examples



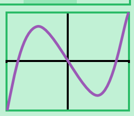
- Key: —— $f(x)$ —— $f'(x)$ - - - Turning Point

Analysing Derivative Graphs Example

(Q1) Sketch the function on the axes below:

x	-2	-1	0	1	2
$f(x)$	$-$	$+$	0	$-$	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f''(x)$	$-$	$-$	0	$+$	$+$

- $x = -2 \rightarrow$ increasing
- $x = -1 \rightarrow$ minimum
- $x = 0 \rightarrow$ vert. inflection
- $x = 1 \rightarrow$ maximum
- $x = 2 \rightarrow$ increasing

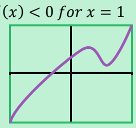


(Q2) Sketch the function on the axes given:

$f(x) \geq 0\ for\ x \geq -1$ $\qquad f''(x) > 0\ for\ x = 2$

$f'(x) = 0\ for\ x = 1, 2$ $\qquad f''(x) < 0\ for\ x = 1$

- $x < 1 \rightarrow$ increasing
- $x = 1 \rightarrow$ maximum
- $x = 2 \rightarrow$ minimum
- $x > 2 \rightarrow$ increasing
- $x \rightarrow \pm\infty, f(x) \rightarrow \pm\infty$



(Q3) Is it possible for a function to have no max or min points but have an inflection point?

- **Yes**, it is possible (e.g. $y = x^3$)
- $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$ at only one point, **(0, 0)**.

INTEGRATION

INTEGRAL LAWS

Indefinite Integrals

$\int x\, dx$	$\int x\, dx = \frac{x^2}{2} + c$

- Indefinite integrals produce an equation and a constant ($+c$) as it caters for a constant in the original function $f(x)$, which disappears (i.e. becomes 0) after being differentiated.

Definite Integrals

$\int_a^b x\, dx$	$\int_1^2 x\, dx = \left[\frac{x^2}{2}\right]_1^2 = \left[\frac{2^2}{2} - \frac{1^2}{2}\right] = 1.5$

- a : integral lower bound (on x-axis).
- b : integral upper bound (on x-axis).

- Definite integrals produce a single number answer (all other variables are eliminated).
- Definite integral of a function that is below the x-axis results in a negative answer.

Common Functions and Integrals

Function	Equation	Integral
Polynomial	$\int x^n dx$	$\frac{x^{n+1}}{n+1} + c$
Chain Rule	$\int f'(x)[f(x)]^n\, dx$	$\frac{[f(x)]^{n+1}}{n+1} + c$
Exponential (Euler)	$\int e^{f(x)}\, dx$	$\frac{e^{f(x)}}{f'(x)} + c$
Reciprocal	$\int \frac{f'(x)}{f(x)}\, dx$	$\ln(f(x)) + c$
Sine	$\int sin(x)\, dx$	$-cos(x) + c$
Cosine	$\int cos(x)\, dx$	$sin(x) + c$

Integration Laws

$\int_a^b f(x) = -\int_b^a f(x)$	$\int_a^a f(x) = 0$

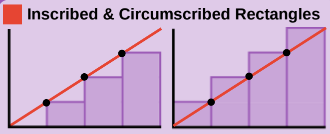
$$\int a \times f(x)\, dx = a \times \int f(x)\, dx$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$\int_b^a f(x) dx + \int_c^b f(x) dx = \int_c^a f(x) dx$$

INTEGRATION BY ESTIMATION

Inscribed & Circumscribed Rectangles



Inscribed	Circumscribed
Series of rectangles below a curve	Series of rectangles above a curve

Underestimation & Overestimation

Underestimation (U)
Adding areas of inscribed rectangles to underestimate area under a curve.

Overestimation (O)
Adding areas of circumscribed rectangles to overestimate area under a curve.

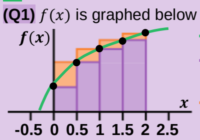
- Overestimation & Underestimation Average:

$\int_a^b f(x) dx \approx \frac{U + O}{2}$	$Area = \sum_i f(x_i)\delta x_i$

- δx : interval size (i.e. width of rectangles).
- U : add the areas of all inscribed rectangles from $x = a$ to $x = b - \delta x$.
- O : add the areas of all circumscribed rectangles from $x = a + \delta x$ to $x = b$.

Estimating Area Under Curve Examples

(Q1) $f(x)$ is graphed below for $-0.5 \leq x \leq 2.5$:



(Q1a) Estimate $\int_0^2 f(x)dx$ using $\delta x = 0.5$:

x	0	0.5	1	1.5	2
$f(x)$	1	2	2.5	2.8	3

$U = 0.5(1 + 2 + 2.5 + 2.8) = 0.5 \times 8.3 = 4.15$

$O = 0.5(2 + 2.5 + 2.8 + 3) = 0.5 \times 10.3 = 5.15$

$Area \approx \frac{U + O}{2} \approx \frac{4.15\ 5.15}{2} \approx \int_0^2 f(x)dx \approx$ **4.65**

(Q1b) If $f(x) = (4x + 1)/(x + 1)$, what is the margin of error in your prediction in part (a)?

$\int_0^2 \frac{4x + 1}{x + 1} dx = 4.7042$ ∴ $4.7042 - 4.65 =$ **0.05**

(Q1c) How can the accuracy of the estimate of the area under curve in part (a)? be increased?

Reduce interval size δx (i.e. smaller than 0.5)

INTEGRATION ALGEBRA

Definite & Indefinite Integral Examples

(Q1) Integrate the function $\int e^{-6x} + 2\sqrt{x} - 4\pi dx$

$2\sqrt{x} = 2x^{\frac{1}{2}}$ ∴ $\int f(x)dx = \frac{-e^{-6x}}{6} + \frac{4x^{\frac{3}{2}}}{3} - 4\pi x + c$

(Q2) Integrate the function $\int 4(3x - 2)^5 dx$

$= \frac{4(3x - 2)^6}{3 \times 6} = \frac{4(3x - 2)^6}{18} = \frac{2(3x - 2)^6}{9} + c$

(Q3) Integrate the function $\int \frac{2x}{1 - 4x^2} dx$

- $\frac{d}{dx}(1 - 4x^2) = -8x$ ∴ numerator must be $-8x$

$-\frac{1}{4}\int \frac{-8x}{1 - 4x^2} dx = -\frac{1}{4}\ln(1 - 4x^2) + c$

(Q4) Integrate the function $\int \frac{1 - 12x^2}{3x} dx$

$\int \frac{1}{3x} - 4x dx = \frac{1}{3}\ln(x) - \frac{4x^2}{2} = \frac{\ln(x)}{3} - 2x^2 + c$

(Q5) Integrate the function $\int 2sin(4 - 3x)dx$

$\frac{-2\cos(4 - 3x)}{-3} = \frac{2\cos(4 - 3x)}{3} + c$

(Q5) Integrate the function $\int 3^{2x} dx$

$\int e^{2ln(3)x} = \frac{e^{2ln(3)x}}{2ln(3)} = \frac{3^{2x}}{2ln(3)} + c$

(Q6) Integrate the function $\int_{-1}^{2}(-e^{3x} + 1)dx$

$= \left[-\frac{e^{3x}}{3} + x\right]_{-1}^{2} = \left[\left(-\frac{e^6}{3} + 2\right) - \left(-\frac{e^{-3}}{3} - 1\right)\right]$

$= -\frac{e^6}{3} + 2 + \frac{e^{-3}}{3} + 1 = \frac{e^{-3} - e^6}{3} + 3$

(Q7) Integrate the function $\int_0^{\pi/4}(2x - cosx)dx$

$= [x^2 - sinx]_0^{\pi/4} = \left[\left(\frac{\pi}{4}\right)^2 - \sin\left(\frac{\pi}{4}\right) - 0\right] = \frac{\pi^2}{16} - \frac{\sqrt{2}}{2}$

Applications of Integration Examples

(Q1) If $\frac{d}{dx}(xe^x) = xe^x + e^x$, determine $\int_0^1 xe^x dx$.

$[xe^x]_0^1 = \int_0^1 xe^x dx + \int_0^1 e^x dx$ $\qquad$ *Create reverse equation and rearrange to find the integral

$\int_0^1 xe^x = [xe^x]_0^1 - \int_0^1 e^x dx$

$\int_0^1 xe^x = [xe^x]_0^1 - [e^x]_0^1$ $\qquad = [e^1 - 0] - [e^1 - e^0]$

$= e - e - -1 =$ **1**

(Q2) If $\frac{dP}{dx} = ax^2 - 12x$ and $P(x)$ has a stationary point at (4,8), determine the value of $P(10)$.

- Use stationary point to determine a:

$0 = 16a - 48 \rightarrow 16a = 48 \rightarrow a = 3$

- Integrate to find $P(x)$ and solve for c:

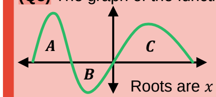
$P = x^3 - 6x^2 + c$ $\qquad 8 = 64 - 96 + c$

$8 = 4^3 - 6(4^2) + c$ $\qquad c = 8 - 64 + 96 = 40$

- Find the value of $P(10)$ using equation:

$P(10) = 10^3 - 6(10^2) + 40 =$ **440**

(Q3) The graph of the function $f(x)$ is shown:



Area $A = 4$ units^2

Area $B = 1$ units^2

Area $C = 3$ units^2

Roots are $x = -10, -5, 0$ & 9.

(Q4a) Determine $\int_{-10}^{9} f(x)dx$

$\int_{-10}^{9} f(x)dx = A - B + c =$ **6**

(Q4b) Determine $\int_0^9 3f(x)dx$

$\int_0^9 3f(x)dx = 3 \times C =$ **9**

(Q4c) Determine $\int_9^{-5} f(x)dx$

$-\int_{-5}^{9} f(x)dx = -C + B =$ **-2**

(Q4d) Determine $\int_{-10}^{-5} f(x) - 2dx$

$= \int_{-10}^{-5} f(x)dx - \int_{-10}^{-5} 2dx$

$= A - [2x]_{-10}^{-5}$

$= 4 - 10 =$ **-6**

AREA UNDER A CURVE

Area Underneath a Curve

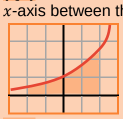
$\int_a^b \lvert f(x) \rvert dx$	$\lvert f(x) \rvert$: absolute value (i.e. change the number inside from negative to positive).

Negative Area Underneath a Curve

Step 1	Determine roots of the function (i.e. factorise and solve for when $y = 0$).
Step 2	Create and add separate integrals that are above and below x-axis.

Area Between Curves Examples

(Q1) Find the area between $y = e^{0.5x}$ and the x-axis between the lines $x = -2$ and $x = 2$.

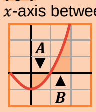


Integral exists above x-axis.

$\int_{-2}^{2} e^{0.5x} dx = [e^{0.5x}]_{-2}^{2}$

$= [e^1 - e^{-1}] =$ **2.35**

(Q2) Find the area between $y = x^2 - 1$ and the x-axis between $x = 0$ and $x = 2$.



$y = x^2 - 1 = (x + 1)(x - 1)$

∴ Root at $x = 1$ which means that integral is above & below x-axis.

∴ Must add 2 separate integrals.

$Area = A + B = \int_1^2 x^2 - 1dx + \int_0^1 |x^2 - 1|dx$

$= [x^2 - 1]_1^2 + [|x^2 - 1|]_0^1 = 3 - 0 + 0 - |-1| =$ **2**

ATAR Math Methods

Units 3 & 4 Exam Notes

Created by **Anthony Bochrinis**

AREA BETWEEN CURVES

Area Between Curves Formulae

- Upper and Lower Bounds on the x-axis:

$$\int_a^b \begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix} dx$$

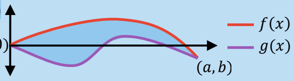
- Upper and Lower Bounds on the y-axis:

$$\int_c^d \begin{pmatrix} right \\ function \end{pmatrix} - \begin{pmatrix} left \\ function \end{pmatrix} dy$$

Area Between Curves Examples

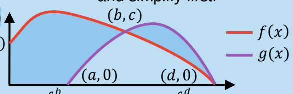
(Q1) Find an expression for finding the shaded area between the two functions $f(x)$ and $g(x)$:

(Q1a)



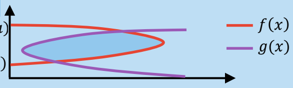
$\int_0^a f(x) - g(x)dx$ *Subtract $g(y)$ from $f(y)$ and simplify first.

(Q1b)



$\int_0^a f(x)dx + \int_a^b f(x) - g(x)dx + \int_b^d g(x) - f(x)dx$

(Q1c)



$\int_b^a f(y) - g(y)dy$ *Rearrange $f(y)$ and $g(y)$ to make y the subject.

(Q2) Find the area between the two different functions $f(x) = \ln(x)$ and $g(x) = (x-4)^2$

- Find intersection points between curves:

$f(x) = g(x)$
$\ln(x) = (x-4)^2$
$x = 2.96, 5.29$

$f(4) = 1.39$ and $g(4) = 0$
$\therefore f(x)$ is upper function and $g(x)$ is lower function.

- Find area between curves:

$\int_a^b upper\ function\, dx - \int_a^b lower\ function\, dx$

$\int_a^b f(x) - g(x)dx = \int_{2.96}^{5.29} \ln(x) - (x-4)^2 dx = \mathbf{2.18}$

RECTILINEAR MOTION

Acceleration/Velocity/Displacement

- Displacement (s): distance from origin.
- Velocity (v): speed toward/away from origin.
- Acceleration (a): rate of change of velocity.

Differentiate

$s \rightleftarrows v \rightleftarrows a$

Antidifferentiate

Rectilinear Motion Formulae

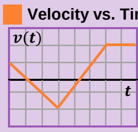
- Terminology of Describing Motion:

Initially	At Origin	Stationary
$t = 0$	$s(t) = 0$	$v(t) = 0$

- Rectilinear Motion Integrals:

Δ Displacement	Distance Travelled
$Change = \int_a^b v(t)\, dt$	$Total = \int_a^b \lvert v(t) \rvert\, dt$

Velocity vs. Time Graphs



- Displacement: area underneath the line.
- Velocity: point on line.
- Acceleration: gradient.

Rectilinear Motion Example

(Q1) Particle X leaves point A, 7 metres from the origin, at a velocity $of\ 4m/s$ at $t = 0$ s and accelerates according to $a = 2t - 4\ m/s^2$.

(Q1a) What is the initial acceleration of X?
$a(0) = 2(0) - 4 = 0 - 4 = \mathbf{-4\ m/s^2}$

(Q1b) When is particle X stationary?
$v = \int a(t)\, dt = \int 2t - 4\, dt = t^2 - 4t + c$
$4 = (0)^2 - 4(0) + c \to 4 = 0 + c \to c = 4$
$\therefore 0 = t^2 - 4t + 4, \therefore$ stationary at $t = \mathbf{2\ secs}$

(Q1c) What is the displacement at $t = 12$ secs?
$s(12) = 7 + \int_0^{12} v(t)dt = 7 + \int_0^{12} t^2 - 4t + 4dt$
$= 7 + \left[\frac{t^3}{3} - 2t^2 + 4t\right]_0^{12} = 7 + 336 = \mathbf{343\ m}$

(Q1d) What is the change of displacement of particle X during the fifth second of motion?
$\int_4^5 t^2 - 4t + 4dt = \left[\frac{t^3}{3} - 2t^2 + 4t\right]_4^5 = \mathbf{6.33\ m}$

(Q1e) What is the maximum speed of particle X during the first 10 seconds of motion?
Solve $\frac{d}{dt}a(t) = 0 \to 2t - 4 = 0, \therefore t = 2$ is a TP
However, $v''(t) = a'(t) = 2$ is a minimum point $\therefore |v(0)| = 4, |v(2)| = 0, |v(10)| = 64$ are the critical points, maximum speed is $\mathbf{64\ m/s}$.

(Q1f) What is the distance travelled by particle X in the first eight seconds of motion?
$d = \int_0^8 |v(t)|dt = \int_0^8 |x^2 - 4x + 4|dt = \mathbf{74.7\ m}$

FINANCIAL CALCULUS

Financial Calculus Terminology

- Marginal Cost ($C'(x)$): cost of producing one additional unit of a product or service.
- Marginal Revenue ($R'(x)$): generated revenue from producing one additional unit.

$R(x) = \int R'(x)dx$	$C(x) = \int C'(x)dx$

- Revenue, Profit and Average Cost:

$\frac{C(x)}{x}$	Average Cost is the cost function divided by x units.
$R(x) = p(x)q(x)$	Revenue is also equal to price multiplied by quantity.
$P(x) = R(x) - C(x)$	Profit is equal to revenue subtract cost for x units.

Financial Calculus Examples

(Q1) The marginal cost of producing x units is $C'(x) = 0.3x^2 - 0.2x + 100$ dollars per unit.

(Q1a) Calculate the extra cost associated with producing the 26th item in the creation of 50:
$C'(25) = 0.3(25)^2 - 0.2(25) + 100 = \mathbf{\$282.50}$

(Q1b) How much more would it cost if 8 units were produced instead of 5 units?
$\int_5^8 C'(x)dx = \int_5^8 0.3x^2 - 0.2x + 100dx = \mathbf{\$334.80}$

(Q1c) If the profit from producing 4 items is \$20 and the marginal revenue function is $R'(x) = x^2$, determine the profit function.
$P'(x) = R'(x) - C'(x) = 0.7x^2 + 0.2x - 100$
$P(x) = \int P'(x)dx = \frac{7x^3}{30} + 0.1x^2 - 100x + c$
$P(4) = 20, \therefore$ solving for $c = 403.67$
$\therefore P(x) = \mathbf{0.7x^2 + 0.2x - 100 + 403.67}$

FUNDAMENTAL THEOREM

Fundamental Theorem of Calculus

$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$	$\int_a^b f(x)\, dx = F(b) - F(a)$

Functions as Integral Limits

Step 1	Substitute the limits into t (only if they are not a constant).
Step 2	Multiply by derivative of the limit. (Note: for questions with two limits, complete steps 1 and 2 twice.)

Fundamental Theorem Examples

(Q1) Determine $\frac{d}{dx}\left(\int_0^x ln(t)dt\right) = \mathbf{ln(x)}$

(Q2) Determine $\frac{d}{dx}\left(\int_0^x e^{2t}dt\right) = \mathbf{e^{2x}}$

(Q3) Determine the derivative $\frac{d}{dx}\left(\int_0^{3x^2} \frac{1+t}{2-t}dt\right)$
$\frac{d(3x^2)}{dx} \times \frac{1+3x^2}{2-3x^2} = 6x\left(\frac{1+3x^2}{2-3x^2}\right) = \mathbf{\frac{6x + 18x^3}{2 - 3x^2}}$

(Q3) Find the derivative $\frac{d}{dx}\left(\int_{sin(x)}^{x^3} \sqrt{t^2+1}\, dt\right)$

- Substituting integral upper limit (*i.e.* x^3):

$\frac{d(x^3)}{dx}\sqrt{(x^3)^2 + 1} = 3x^2\sqrt{x^6+1}$

- Substituting integral lower limit (*i.e.* $sinx$):

$\frac{d(sinx)}{dx}\sqrt{(sinx)^2+1} = \cos(x)\sqrt{(\sin(x))^2+1}$

- Subtract 2nd answer from 1st answer:

$= \mathbf{3x^2\sqrt{x^6+1} - \cos(x)\sqrt{(\sin(x))^2+1}}$

(Q4) Find $f(x)$ with the following conditions:
$F(x) = \int_0^x f(t)dt,\ \frac{d^2F}{dx^2} = x + 5,\ F(3) = 5$:
$\frac{dF}{dx} = f(x)$, Hence $\frac{d^2F}{dx^2} = f'(x) = x + 5$

- Integrating $f'(x)$ to get $f(x)$:

$f(x) = \int x + 5\, dx = \frac{x^2}{2} + 5x + c$

- Use the $F(x)$ formula to solve for c:

$F(3) = \int_0^3 f(t)dt = 5\ \ \therefore c = -7.33$
$5 = \int_0^3 \frac{t^2}{2} + 5t + c\, dt\ \ \therefore f(x) = \mathbf{\frac{t^2}{2} + 5t - 7.33}$

(Q5) Find $f(x)$ with the following conditions:
$F(x) = \int_0^x f(t)dt,\ \frac{d^2F}{dx^2} = x^2,\ f(2) = 2$:
$\frac{dF}{dx} = \frac{d}{dx}\int_0^x f(t)dt = f(x), \frac{d^2F}{dx^2} = f'(x) = x^2$
$\therefore f(x) = \frac{x^3}{3} + 3 \to if\ f(2) = 2, then\ 2 = \frac{8}{3} + c$
$\frac{6}{3} = \frac{8}{3} + c \to c = -\frac{2}{3}, \therefore f(x) = \frac{x^3}{3} + c = \mathbf{\frac{x^3}{3} - \frac{2}{3}}$

(Q6) $f(x)$ is increasing on interval $0 < x < 3$ and decreasing on $3 < x < 6$ as per the table:

x	0	1	2	3	4	5	6
$f(x)$	5	16	27	32	25	0	−49

Let $F(x) = \int_0^x f(t)dt$ on interval $0 \le x \le 6$.

(Q6a) What value of x is $F(x)$ the greatest?
$F(x)$ is the area under the graph of $f(x)$, so when $f(x) > 0$ gives greatest area $\therefore \mathbf{x = 5}$

(Q6b) What value of x is $F'(x)$ the greatest?
$F'(x) = f(x) \therefore$ greatest is maximum $\therefore \mathbf{x = 3}$

PROBABILITY

SET NOTATION

Logic Functions and Symbols

- $\bar{A}$ or A': complement of an event (not A).
- $A \cup B$: union of two events (A or B).
- $A \cap B$: intersection of two events (A and B).

Notation and Symbols

- $\in$: element (found in a given set).
- $\notin$: not an element (not found in a given set).
- $\emptyset$ or { }: empty set (contains no elements).
- $\mathbb{U}$: universal set (contains all elements).
- $\subset$: subset ($A \subset B$ means that all elements of set A is found within the elements of set B).
- $n(A)$ or $|A|$: number of elements in set A.

PROBABILITY RULES

Probability Laws

- Rule of Subtraction (*i.e.* not A):

$$P(\bar{A}) = P(A') = 1 - P(A)$$

- Rule of Addition (*i.e.* A or B):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Rule of Multiplication (*i.e.* A and B)

$P(A \cap B) = P(A) \times P(B \vert A)$	$P(A \cap B) = P(B) \times P(A \vert B)$

- Conditional Probability (*i.e.* A given B)

$P(A \vert B) = \frac{P(A \cap B)}{P(B)}$	$P(B \vert A) = \frac{P(A \cap B)}{P(A)}$

Conditional Probability Terminology

- $P(A|B)$ means the probability of A occurring given that B has already occurred.

Mutually Exclusive Events

Rule 1	$P(A \cap B) = 0$
Rule 2	$P(A \cup B) = P(A) + P(B)$

Independent Events

Rule 1	$P(A \cap B) = P(A) \times P(B)$	
Rule 2	$P(A \vert B) = P(A)$	$P(B \vert A) = P(B)$

RANDOM VARIABLES

EXPECTED VALUE & VARIANCE

Probability Notation

Expected Value	Variance	Standard Deviation
$E(X) = \mu$	$Var(X) = \sigma^2$	$\sqrt{Var(X)} = \sigma$

- Relationship between $E(X)$ and $Var(X)$:

$$Var(X) = E(X^2) - [E(X)]^2$$

Effects of Linear Change

- If X is random variable and $Y = aX + b$ then:

$E(Y) = aE(X) + b$	$Var(Y) = a^2Var(X)$

- a and b : constants (*i.e.* numbers).

Effects of Linear Change Examples

(Q1) If $E(X) = 5$ and $Var(X) = 2$, determine:
(Q1a) $E(X + 11) = E(X) + 11 = 5 + 11 = \mathbf{16}$
(Q1b) $E(1 - 2x) = 1 - 2E(X) = 1 - 10 = \mathbf{-9}$
(Q1c) $Var(3X + 1) = 3^2Var(X) = 9 \times 2 = \mathbf{18}$

DISCRETE RANDOM VARIABLES

Discrete Random Variables *(DRV)*

- Discrete distributions are events that can be counted in integers (*i.e.* whole numbers).
- Types of DRV's: Bernoulli and Binomial.

DRV Rules and Notation

$\sum P(X = x) = 1$	$0 \le P(X = x) \le 1$
$P(X = a)$ can be calculated	
$P(X < a) = P(X \le a - 1)$	$P(X > a) = P(X \ge a + 1)$

$E(X)$	$Var(X)$
$\sum xp(x)$	$\sum (x - \mu)^2 p(x)$

Discrete Random Variable Examples

(Q1) Is the following distribution discrete?

x	-1	0	1	2
$P(X = x)$	0.3	0.2	0.1	0.4

$0.3 + 0.2 + 0.1 + 0.4 = 1 \therefore$ **Yes**, this is a DRV.

(Q2) Is the following distribution discrete?

x	0	1	2	3
$P(X = x)$	-0.1	0	0.5	0.6

$P(X = 0) =$ negative $\therefore$ **No**, not a DRV.

► Topic Is Continued In Next Column ◄

DISCRETE RANDOM VARIABLES

Discrete Random Variable Examples

(Q3) Determine the values of a and b in the following discrete distribution if $E(X) = 0.20$:

x	0	1	2	3	4
$P(X = x)$	0.85	0.12	a	b	0.005

Equation 1: $0.12 + 2a + 3b + 0.2 = 0.2$
Equation 2: $0.85 + 0.12 + a + b + 0.005 = 1$
Simultaneously solve: $a = \mathbf{0.015}$ and $b = \mathbf{0.01}$

(Q4) Probability distribution for X is shown:

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.2	0.3	0.2

(Q4a) Calculate the value of $Var(X)$:
$E(X) = 1(0.1) + 2(0.2) + \cdots + 5(0.2) = 3.3$
$E(X^2) = 1^2(0.1) + 2^2(0.2) + \cdots + 5^2(0.2) = 12.5$
$Var(X) = E(X^2) - [E(X)]^2 = 12.5 - 3.3^2 = \mathbf{1.61}$

(Q4b) Find the cumulative distribution for X:

x	1	2	3	4	5
$P(X \le x)$	0.1	0.3	0.5	0.8	1

(Q4c) Find the probability $P(X > 2|X \le 4)$:
$\frac{P(X \ge 3) \cap P(X \le 4)}{P(X \le 4)} = \frac{0.2 + 0.3}{0.8} = \frac{0.5}{0.8} = \mathbf{0.625}$

BERNOULLI DISTRIBUTION

Bernoulli Distribution and Notation

- Only two possible outcomes; either "success" or "failure" and are independent of other trials.
  - *e.g.* tossing a coin: either heads or tails.

$X \sim Ber(p)$	• p : probability of success

Bernoulli Distribution Rules

$$P(X = x) = \begin{cases} p & for\ x = 1 \\ 1 - p & for\ x = 0 \end{cases}$$

$E(X) = \mu$	$Var(X) = \sigma^2$	$S.D. = \sigma$
p	$p(1 - p)$	$\sqrt{p(1 - p)}$

BINOMIAL DISTRIBUTION

Binomial Distribution and Notation

- A binomial trial is more than one bernoulli trial.
- Counts the number of successes in an independent number of trials.
  - *e.g.* tossing a coin repeat times and counting the number of heads flipped.

$X \sim Bin(n, p)$	• n : number of trials • p : probability of success

Binomial Distribution Rules

$$P(X = x) = \binom{n}{x}(p)^x(1 - p)^{n-x}$$

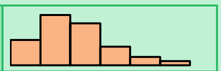
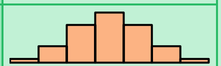
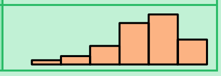
$${}^nC_r = n\ choose\ r = \binom{n}{r} = \frac{n!}{(n - r)! \times r!}$$

$E(X) = \mu$	$Var(X) = \sigma^2$	$S.D. = \sigma$
np	$np(1 - p)$	$\sqrt{np(1 - p)}$

Binomial Calculator Notation

$P(X)$	$binomialPDF(x, n, p)$
$P(A \le X \le B)$	$binomialCDF(A, B, n, p)$
$P(X \le k)$	$invBinCDF(P(X \le k), n, p)$

Graphing Binomial Distributions

$p < 0.5$ Skewed right (positive)	
$p = 0.5$ Symmetrical	
$p > 0.5$ Skewed left (negative)	

APPLICATIONS OF DRV'S

Bernoulli and Binomial DRV Examples

(Q1a) Find chance that Ben passes multi-choice test with 10 Q's with 4 options each by guessing?
$X \sim Bin(10, 0.25) \to P(5 \le X \le 10) = \mathbf{0.0781}$

(Q1b) If the class has 15 students, what is the probability that at least 4 pass by guessing?
$X \sim Bin(15, 0.0781) \to = P(4 \le X \le 15) = \mathbf{0.025}$

(Q2a) Chance of a rotten apple is 0.1. Find the chance that 1 apple is rotten out of 6 apples.
$P(1\ Rotten\ Apple) = 0.9^5 \times 0.1 = \mathbf{0.0590}$

(Q2b) Find the chance that exactly one of six apples randomly chosen are found to be rotten.
$X \sim Bin(6, 0.1) \to P(X = 1) = \mathbf{0.3543}$

► Topic Is Continued On Next Page ◄

APPLICATIONS OF DRV'S

Bernoulli and Binomial DRV Examples

(Q3a) X is binomial variable. Find the value of n and p if $E(X) = 21$ and $Var(X) = 6.3$.

$E(X) = 21 = np$ & $Var(X) = 6.3 = np(1-p)$

Simultaneously solve: $n = \mathbf{30}$ and $p = \mathbf{0.7}$

(Q3b) Find the probability $P(X \geq 10 | X \leq 15)$:

$$\frac{P(X \geq 10 \cap X \leq 15)}{P(X \leq 15)} = \frac{P(10 \leq X \leq 15)}{P(X \leq 15)}$$

$$X \sim B(30, 0.7),\ \frac{binPDF(10,15,30,0.7)}{binPDF(0,15,30,0.7)} = \mathbf{0.9996}$$

(Q4) Find the probability of rolling a 5 at least two times on a 6-sided dice from ten throws.

$X \sim Bin(10, 0.167) \to P(X \geq 2) = 1 - P(X = 0)$
$-P(X = 1) = 1 - 0.1609 - 0.3225 = \mathbf{0.5166}$

(Q5) The chance of success is 0.4, how many trials are needed to ensure that the probability of 3 or more successes is exceeds 0.75?

$X \sim B(n, 0.4)$ and requirement $P(X \geq 3) > 0.75$

- Trial and error for different values of n

$binCDF(3, \infty, n, 0.4), n = 9, CDF = 0.7682 \ \therefore \mathbf{9}$

(Q6) A game store charges \$3 to play a game. Two dice are rolled and the uppermost faces are added with the prizes being as follows:

Sum	7	3 or 5	9 or 11	Even
Payout	\$0	\$4	\$6	\$1

Is this game expected to be profitable?

Sum	7	3 or 5	9 or 11	Even
Profit	\$3	-\$1	-\$3	\$2
Prob.	1/6	1/6	1/6	1/2

$$E(X) = 3\left(\frac{1}{6}\right) - 1\left(\frac{1}{6}\right) - 3\left(\frac{1}{6}\right) + 2\left(\frac{1}{2}\right) = \$\mathbf{0.83}$$

$\therefore$ at \$3 per game, expected to profit \$0.83.

CONTINUOUS RANDOM VARIABLES

Continuous Random Variables *(CRV)*

- Continuous distributions are events that can be measured in decimal numbers.
- Types of XRV's: Uniform and Normal.

CRV Rules and Notation

$\int p(x)dx = 1$	$0 \leq P(X = x) \leq 1$
$P(X = a)$ cannot be calculated	
$P(X < a) = P(X \leq a)$	$P(X > a) = P(X \geq a)$

$E(X)$	$Var(X)$
$\int_{-\infty}^{\infty} xp(x)dx$	$\int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx$

Discrete Random Variable Examples

(Q1) X is a CRV given that $P(X > 5) = 0.6$ and X has a probability density function of:

$$f(x) = \begin{cases} ax + b & 0 \leq x \leq 10 \\ 0 & elsewhere \end{cases}$$ Find a and b:

Equation 1: $\int_0^{10} ax + b\, dx = 1$ *Sums to 1

Equation 2: $\int_5^{10} ax + b\, dx = 0.6$ *Given in Q.

Simultaneously solve: $a = \mathbf{0.008}$ & $b = \mathbf{0.06}$

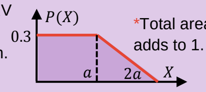
(Q2) Y is a CRV with a density function of:

$$f(y) = \begin{cases} 2y^2 + 3 & 0 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$$ Find $E(Y)$ and $Var(Y)$:

$$E(Y) = \int_0^2 (y)(2y^2 + 3)dy = \mathbf{14}$$

$$Var(Y) = \int_0^2 (y - 14)^2(2y^2 + 3)dy = \mathbf{1850.13}$$

(Q3) X is a CRV with a density function shown. Determine a.



*Total area adds to 1.

$1 = 0.3a + 0.5(0.3a) \to 1 = 0.45a \to a = \mathbf{2.22}$

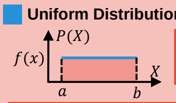
UNIFORM DISTRIBUTION

Uniform Distribution and Notation

- Uniform distribution has constant probability.
  - *e.g.* a volcano erupts randomly every hour.

$X \sim U(a, b)$	• a : lower boundary • b : upper boundary

Uniform Distribution Rules



$$f(x) = \frac{1}{b-a}$$

$$Area = l \times w = (b - a) \times \frac{1}{b-a} = 1$$

$E(X) = \mu$	$Var(X) = \sigma^2$	$S.D. = \sigma$
$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\sqrt{\frac{1}{12}(b-a)^2}$

$P(c \leq X < d)$	Find k given $P(X \leq k)$
$\int_c^d \frac{1}{b-a}dx$	$\int_a^k \frac{1}{b-a}dx = P(X \leq k)$

► Topic Is Continued In Next Column ◄

UNIFORM DISTRIBUTION

Uniform Distribution Examples

(Q1) X is uniform with $a = 10$ and $b = 20$.

(Q1a) Determine the value of $P(X \geq 14)$:

$X \sim U(10, 20) \to \int_{14}^{20} \frac{1}{20-10}dx = \mathbf{0.6}$

(Q1b) Determine $P(X \geq 14 | X \leq 18)$:

$\frac{P(14 \leq X \leq 18)}{P(X \leq 18)} = \int_{14}^{18} \frac{1}{20-10}dx \div \int_{10}^{18} \frac{1}{20-10}dx = \mathbf{0.5}$

(Q2) Y is uniform with $a = 1$ and $b = 5$.

(Q2a) Find k given $P(X > k | X < 3) = 0.5$:

$\frac{P(k<X<3)}{P(X<3)} = 0.5 \to P(k < X < 3) = 0.25 \ \therefore k = \mathbf{2}$

(Q2b) Find k given $P(X > 2 | X < k) = 0.5$:

$\frac{P(2<X<k)}{P(X<k)} = 0.5 \to P(2 < X < k) = 0.5P(X < k)$

Using trial and error for values of k: $k = \mathbf{3}$

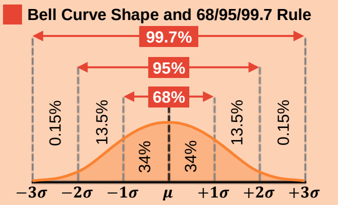
NORMAL DISTRIBUTION

Normal Distribution *(Bell Curve)*

- Has greater probability closer to the mean.
  - *e.g.* average test scores for a whole class.

$X \sim N(\mu, \sigma^2)$	• μ : mean • σ^2 : variance

Bell Curve Shape and 68/95/99.7 Rule



- 50% of all scores are above the mean and 50% of all scores are below the mean.

Z-Scores *(Standardised Scores)*

- Simplifies all normal distributions to a mean of 0 and a standard deviation of 1.
- Indicates how many standard deviations away from the mean each score is.

$Z \sim N(0, 1^2)$	$z = \frac{x - \mu}{\sigma}$

Distribution Percentiles

- a% of data lies below the a^{th} percentile.

$P(X < k_a) = a$	• a : percentile $0 < a < 1$

Normal Distribution Calculator Notation

$P(A \leq X \leq B)$	$normCDF(A, B, \sigma, \mu)$
Find k given $P(X \leq k)$	$invNormCDF$ $(Tail\ Setting, P(X \leq k), \sigma, \mu)$

Left Tail Centered Right Tail

Normal Distribution Examples

(Q1a) $X \sim N(20, 5^2)$, find x with a z-score of 1.5:

$z = \frac{x - \mu}{\sigma} \to 1.5 = \frac{x - 20}{5} \to x = \mathbf{27.5}$

(Q1b) Find the value of the 67^{th} percentile:

Find the value of k such that $P(X < k) = 0.67$:

$k = invNormCDF(Left, 0.67, 5, 20) = \mathbf{22.20}$

(Q1c) Find the value of $P(X < 21 | X > 16)$:

$$\frac{P(16 < X < 21)}{P(X > 16)} = \frac{nCDF(16, 21, 5, 20)}{nCDF(16, \infty, 5, 20)} = \mathbf{0.466}$$

(Q1d) Find the value of k for $P(X > k) = 0.75$

$k = invNormCDF(Right, 0.75, 5, 20) = \mathbf{16.63}$

(Q2) If $X \sim N(\mu, \sigma^2)$ and the mean is twice the variance and $P(X > 10) = 0.3$. Find μ and σ.

$\mu = 2\sigma^2, \therefore X \sim N(2\sigma^2, \sigma^2)$.

invNormCDF("L", 0.3, 1, 0) = 0.5244 *Solve with Z-Scores

$Z = \frac{X-\mu}{\sigma} \to 0.5244 = \frac{10-\mu}{\sigma} = \frac{10-2\sigma^2}{\sigma}$

Solve $0.5244 = \frac{10-2\sigma^2}{\sigma}$, $\sigma = \mathbf{2.11}$ and $\mu = \mathbf{8.89}$

INTERVAL ESTIMATES

RANDOM SAMPLING

Impact of Bias on Samples

- If survey is biased, sample stats will not reflect population stats (*i.e.* sampling error).

Types of Sampling Bias

- Selection Bias: issues with sampling.
  - Undercoverage: when members of the population aren't adequately represented.
  - Nonresponse: views of non-respondants are missed as they are unwilling and/or unable to participate in the survey.
  - Voluntary Response: sampling people who will only willingly participate.
- Response Bias: issues with surveying.
  - Leading Question: persuades a response.
  - Loaded Question: too much information.

Methods of Reducing Sampling Error

- Increase the sample size.
- Exercise true random sampling methods:
  - Systematic: select every n^{th} person/item.
  - Stratified: sample groups that reflect size of same groups in entire population.

CENTRAL LIMIT THEOREM

Central Limit Theorem *(CLT)*

- If there are a large number of independent random samples (*i.e.* $n \geq 30$), the data can be modelled using a normal distribution.
- Also appropriate if np and $np(1-p) \geq 10$.
- Uses sample size not number of samples.

CLT of a Random Variable X

- μ is population mean and $\bar{X}$ is sample mean.
- If $n \geq 30$, $X \sim N$ with the following parameters:

Mean *(stays)*	S.D. *(changes)*	Z-Score *(changes)*
$\bar{X}$	$\frac{\sigma}{\sqrt{n}}$	$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

CLT Special Case: Bernoulli Distribution

- p is population mean and $\hat{p}$ is sample mean.
- If $n \geq 30$, $X \sim N$ with the following parameters:

Mean *(stays)*	S.D. *(changes)*	Z-Score *(changes)*
$\hat{p}$	$\sqrt{\frac{p(1-p)}{n}}$	$Z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$

Central Limit Theorem Examples

(Q1a) 18% of pizzas are overcooked at a store. In a sample of 150 pizzas, find the distribution.

$\mu = \hat{p} = 0.18$, $\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.18(1-0.18)}{150}} = 0.03$

Hence, $\mathbf{X \sim N(0.18, 0.0314^2)}$

(Q1b) What is the chance that the proportion of overcooked pizzas exceeds 0.21.

$X \sim N(0.18, 0.0314^2) \to P(X > 0.21) = \mathbf{0.1697}$

(Q2a) 23% of Australians are left handed. If 40 are surveyed, what proportion of samples are expected to have less than 20% left-handers?

$\mu = \hat{p} = 0.23$, $\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.23(1-0.23)}{40}}$

$X \sim N(0.23, 0.0665^2)$ and $P(X < 0.2) = \mathbf{0.3257}$

(Q2b) What proportion of samples will expect to contain between 10% and 15% left-handers?

$P(0.10 < X < 0.15) = \mathbf{0.0892}$

(Q3) The wait for pedestrians at traffic lights is uniformly distributed between 0 and 4 mins.

(Q3a) The waiting times of 20 samples of 45 pedestrians are recorded, find the distribution:

- Find μ and σ of the uniform distribution:

$\mu = 0.5(a + b) = 0.5(0 + 4) = 2$

$\sigma = \sqrt{(b-a)^2/12} = \sqrt{16/12} = 1.1547$

- Change σ according to the CLT rule:

$\frac{\sigma}{\sqrt{n}} = \frac{1.1547}{\sqrt{45}} = 0.1721 \therefore \mathbf{X \sim N(2, 0.1721^2)}$

(Q3b) Find the chance that a random sample has a mean time of less than 1.9 minutes:

$X \sim N(2, 0.1721^2) \to P(X < 3) = \mathbf{0.2806}$

(Q3c) What is the chance that at most 5 samples had mean time of less than 1.9 mins:

$X \sim Bin(20, 0.2806) \to P(X \leq 5) = \mathbf{0.4928}$

(Q4a) Out of a sample size of 53, it was found that 70% of teachers have a computer at home. Is it reasonable to use the normal distribution to approximate this sample proportion?

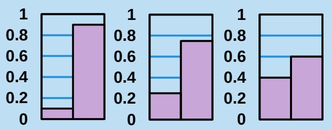
$n \geq \mathbf{30}, np = 37.1 \geq \mathbf{10}, n(1-p) = 15.9 \geq \mathbf{10}$

$\therefore$ sample size and proportion is large enough to approximate as a normal distribution.

(Q4b) Determine the parameters of the normal distribution that it would approximate to:

$\mu = \mathbf{0.7}$, $\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.7(1-0.7)}{53}} = \mathbf{0.0629}$

(Q5) A Bernoulli distribution has a probability of success of 0.6. Random samples of size 20, 50 and 500 were taken and plotted on the following relative frequency graphs:



(Q5a) Match the graphs with the sample sizes:

- **First** graph is sample size of **20** (as it has the highest variability from success prob.).
- **Second** graph is sample size of **50**.
- **Third** graph is sample of size **500** (as it closely matches probability of success).

(Q5b) 100 random samples are taken from this distribution by recording number of successes. Describe the features of a frequency graph that shows distribution of proportion of successes by repeating a large number of times.

Graph to be **normally distributed** with:

$\mu = \mathbf{0.6}$, $\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.6(1-0.6)}{100}} = \mathbf{0.04899}$

(Q6) 100 samples of 10 people are tested for a disease (prob. = 0.005) Also, 100 samples of 100 people are tested for same disease. Which set resembles normal distribution the most?

Second set of samples, as CLT uses sample size not number of samples (*i.e.* $100 > 10$).

CONFIDENCE AND ERROR MARGINS

Confidence Intervals *(CI)*

- Probability that confidence interval (at a certain level) will contain the population proportion.

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = (CI_L, CI_U)$$

- Z : z-score for a given confidence interval.
- CI_L : confidence interval lower bound.
- CI_U : confidence interval upper bound.

- Commonly used Confidence Intervals:

% Confidence Interval	Z-Score
99% Confidence Interval	2.58
95% Confidence Interval	1.96
90% Confidence Interval	1.645

- Custom Confidence Intervals:

$$z_{CI\%} = -1 \times invNormCDf("C", CI\%, 1, 0)$$

- Z : z-score for a given confidence interval.
- CI_L : confidence interval lower bound.
- CI_U : confidence interval upper bound.

Margin of Error *(E)*

- The margin of error is half the width of a confidence interval (*i.e.* $Total\ Width = 2E$).
- Maximum difference between $\hat{p}$ and $z\sigma$.

$E = z\sigma = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	$E \propto z$	$E \propto \frac{1}{\sqrt{n}}$
$CI = \hat{p} \pm E$	$p = \frac{CI_L + CI_U}{2}$	$E = \frac{CI_U - CI_L}{2}$

Changing Confidence Intervals

Step 1	Use CI bounds to find the value of p: $p = (CI_L + CI_U)/2$
Step 2	Determine margin of error E: $E = CI_U - p$ or $E = p + CI_L$
Step 3	Calculate: $E_{new} = z_{new}/z_{old} \times E_{old}$
Step 4	Determine new confidence interval: $New\ CI = p \pm E_{new}$

Interval Examples

(Q1) A 90% confidence interval is $(0.38, 0.45)$. Determine the 95% confidence interval.

$p = \frac{0.38 + 0.45}{2} = 0.415$ $\quad E_{new} = \frac{1.96}{1.645} \times 0.035$

$E = 0.45 - 0.415 = 0.035$ $\quad = 0.0417$

$95\%\ CI = 0.415 \pm 0.0417 = \mathbf{(0.3733, 0.4567)}$

(Q2) How many times larger is margin of error of sample of 1225 compared to sample of 11025?

$E \propto \frac{1}{\sqrt{1225}} = \frac{1}{35}, E \propto \frac{1}{\sqrt{11025}} = \frac{1}{105} \therefore \mathbf{3}$ times larger.

(Q3) Find sample size of a survey with sample proportion of 0.6 for a 99% CI with $E = 0.02$.

$E = z\sigma \to 0.02 = 2.58\sqrt{0.6 \times 0.4/n} \to n = \mathbf{3982}$

(Q4) In a random sample of 400 people, 129 we male. Calculate a 90% confidence interval.

$p = 0.3225 \to CI = 0.3225 \pm 1.645\sqrt{\frac{0.3225 \times 0.6675}{400}}$

$90\%\ CI = \mathbf{(0.2898, 0.3552)}$

(Q5) What is the margin of error on a 99% confidence interval of (0.25,0.32)?

$E = \frac{CI_U - CI_L}{2} = \frac{0.32 - 0.25}{2} = \mathbf{0.035}$

(Q6) 28% of residents in a city are aged 20 or below. 300 samples of 100 residents were taken. How many S.D.'s below the pop. proportion will a sample of 21 of 100 residents be 20 or under?

$p = 0.28$, $\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.28(1-0.28)}{100}} = 0.0449$

$Z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} = \frac{0.21 - 0.28}{0.0449} = \mathbf{-1.559}\ S.D.$

(Q7) 30 out of 150 students at a school hate chocolate. Find the 95% CI and interpret it.

$p = \frac{30}{150} = 0.2$, $\sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2(1-0.2)}{150}} = 0.0327$

$z = 1.96, 95\%\ CI = (0.2 \pm 1.96 \times 0.0327)$

$95\%\ CI = \mathbf{(0.1359, 0.2641)}$ which means 95% confident that the population proportion lies between 0.1359 and 0.2641 (*i.e.* 13.59% to 26.41% of students at a school hate chocolate).

(Q8) A 90% CI for a sample of students who ordered lunch at the canteen today is (0.12,0.46). Find how many in the sample ordered lunch.

$p = \frac{0.12 + 0.46}{2} = 0.29$, $E = \frac{0.46 - 0.12}{2} = 0.17$

$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \to 0.17 = 1.645\sqrt{\frac{0.29(1-0.29)}{n}}$

Solve for $n = 20 \therefore 0.29 \times 20 = \mathbf{6}$ students